THE GAMBLER'S RUIN

By J. L. Coolidge

The various problems which are connected with games of chance have interested mathematicians ever since the time of Pascal. In particular, the probability that when two men are playing together at a certain game, the one will end by ruining the other, and the probable number of turns before one player's ruin, has been studied by James Bernouilli, DeMoivre, Lagrange, Laplace, and others.* In all of these works, however, it is assumed that the amount of the stakes is the same at each turn, and the problem of showing that, regardless of the amount which is staked, the probability of ruin for the one or the other player, lies within certain definite limits, has not, so far as I have been able to discover, ever been undertaken. In general the mathematician assumes, without any proof, that whoever indulges in a game favorable to his adversary has a very strong chance of being ruined, regardless of what system of play he may affect. The player, on the other hand, ascribing to the goddess of fortune a desire which she does not possess, to atone for past errors, works out a system of play in which he confides. It is the object of the present paper to show that no such system can have more than an easily calculable chance of success.

1. Unlimited Stakes. Let us imagine that two persons called respectively the "player" and the "banker" are playing at a game of chance. The effective difference between the two is that the player has the right to settle the amount of the stake each time; and also to leave off when he chooses. Suppose, first, that the player has a capital which, compared with the banker's, may be roughly called indefinitely great. Such is, in fact, approximately the case when the banker conducts a public gaming establishment ready to meet all comers. The probability that the player will be ruined is, here, negligible. On the other hand if the player begin by staking the amount of the banker's fortune and doubles his stake every time he loses, the probability that he will end by ruining the banker, is very large indeed. For that reason public gaming establishments habitually set an upper limit to the amount which

* See Todhunter's *History of the Theory of Probability*, London, 1865.

may be staked. Let us, secondly, suppose that whereas the player has only a fortune of m the banker's fortune is quasi-infinite in comparison therewith. Let us say that a first game is finished when either the player is ruined, or he has doubled his original fortune. As in this game his stake is surely less than $2m$ we are under the second case of limited stakes, to be treated presently, and the chance that the player will escape ruin is not over one-half; unless the game is favorable to him, as is not the case in public gaming. If the player win the first game, let him start another which shall terminate when the one side or the other has won $2m$. Once more his chance of escaping ruin is $\frac{1}{2}$. Keeping on thus we see that if the player undertake to play an indefinite number of games, the chance that he will never be ruined is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \cdots$; indefinitely small.

If the player and banker have each a finite sum, the maximum stake is surely less than their combined fortunes, and we are under our second case.

2. Limited Stakes. Let us assume that the player has a fortune m and that he undertakes to play until either he losses the whole of this, or wins the sum n from the banker. This is the usual state of affairs in public gambling, although some players being with a very indefinite notion of the value of n, which has the effect of making n very large, thereby greatly increasing the probability that the player will be ruined. When the player has won the sum n we shall, for convenience, say that the bank has been ruined, though the player of ordinary means who should really set n at the amount of the bank's funds, would do a very foolish thing. For the sake of simplicity we shall assume that if the player is so nearly ruined that he cannot afford to stake as much as he wishes, then he stakes all that he has, whereas if his winnings are so near to n that the residue is less than the banker should put up in order to cover the player's usual stake, then also we shall assume that the player reduces his stake accordingly. As a matter of fact it does not make any great difference how we treat these extreme circumstances; unless $m + n$ is a comparatively small multiple of the amount of the stake, or unless the game is grossly unfavorable to one side; when either of these cases arises, the ruin of the less favored party is almost certain.

What do we mean by the game's being favorable or unfavorable to one side?

If a man have mutually exclusive chances $p_1 p_2 \cdots p_r$ where $\sum_{i}^{1=r} p_i = 1$

of obtaining the positive or negative values $a_1\, a_2 \cdots a_r$ respectively, then the expression

$$\sum_{i=1}^{i=r} p_i a_i$$

is defined as his *expectation*. An individual play, or a whole game, is said to be *fair* when the expectation of each person before entering is zero. Thus, let the player stake a with the probability p of winning, while the banker, whose favorable chance is $q = 1 - p$, replies with a stake b. The player's expectation is $pb - qa$, while that of the banker is $qa - pb$. The criterion for a fair play is $pb - qa = 0$,

or $$\frac{a}{p} = \frac{b}{q}. \tag{1}$$

In a fair play between two persons, the stake of each is proportional to the probability that he will win.

This rule is easily extended to any number of persons.

Suppose, next, that we have a succession of two fair plays, the player's and banker's chances and stakes being respectively, a_1, p_1; b_1, q_1 in the first, a_2, p_2; b_2, q_2 in the second. The player's expectation from the game composed of these two is

$$\begin{aligned} p_1 p_2(b_1 + b_2) + p_1 q_2(b_1 - a_2) + p_2 q_1(b_2 - a_1) - q_1 q_2(a_1 + a_2) \\ = (p_1 + q_1)(p_2 b_2 - q_2 a_2) + (p_2 + q_2)(p_1 b_1 - q_1 a_1) = 0. \end{aligned}$$

This reasoning may be continued indefinitely, whence

Any succession of fair plays will constitute a fair game.

The commonest games are those where p and q retain constant values; let us show that if such a fair game be continued indefinitely the ruin of either player or banker is certain. The player's fortune is set at m, the banker's at n. We first suppose that the stakes are constant, a and b. If v turns be played, the probability that the player shall win not more than $vp + k$ nor less than $vp - k$ is, by Bernouilli's theorem,

$$\frac{2}{\sqrt{\pi}}\int_0^{\frac{k}{\sqrt{2vpq}}} e^{-t^2}\, dt$$

and for any constant value of k, this will approach zero as a limit as v increases indefinitely. Replacing k by κ, which is supposed to be either positive or negative, if the player win $vp + \kappa$ turns his winnings will be

$$(vp + \kappa)\, b - (vq - \kappa)\, a = \kappa(a + b).$$

Take κ numerically larger than the larger of the two quantities $m/(a + b)$, $n/(a + b)$, and allow the play to continue indefinitely; the ruin of the one or other party is seen to be certain.

The case where the player may alter his stake at will requires more delicate handling. Let us suppose that in v turns the player makes the stake a_1, r_1 times, in general the stake, a_i, r_i times; the banker's corresponding stakes being $b_1, b_2, \cdots b_r$.

We have
$$\sum_{i=1}^{i=r} r_i = v.$$

The player's chance of winning a particular play is independent of the amount staked, so that of the $vp + \kappa$ plays which he wins, the most probable number to be associated with the stake, a_i, will be $(vp + \kappa)\,\frac{r_i}{v}$. Let us assume, therefore that, of the a_i play he wins $r_i p + \frac{\kappa r_i}{v} + \epsilon_i$. His total winnings will be

$$\sum_{i=1}^{i=r}\left[\left(p + \frac{\kappa}{v}\right)r_i + \epsilon_i\right]b_i - \sum_{i=1}^{i=r}\left[\left(q - \frac{\kappa}{v}\right)r_i - \epsilon_i\right]a_i$$
$$= \sum_{i=1}^{i=r}\left(\frac{\kappa}{v}\, r_i + \epsilon_i\right)\left(a_i + b_i\right).$$

Let us modify this slightly. Since $vp + \kappa$ turns are won,

$$\sum_{i=1}^{i=r} \epsilon_i = 0.$$

If, then a_1 be the smallest stake, we may write the player's winnings as

$$\frac{\kappa}{v}\sum_{i=1}^{i=r} r_i(a_i + b_1) + \sum_{i=2}^{i=r} \epsilon_i[(a_i - a_1) + (b_i - b_1)].$$

The quantity κ is, by Bernouilli's theorem, equally likely to be positive or negative. If, after v_1 turns, it be found to be positive, after v_2 turns more it will have a greater chance of being positive than negative, but this disproportion approaches zero as a limit, as v_2 increases indefinitely, and may be disregarded when v_2 is very large. The quantities $\epsilon_2 \cdots \epsilon_r$ may be positive or negative, and may all have like signs, or opposite signs in varying proportions. Let us assume, hence, that P represents the probability that after v_j turns, the sign of ϵ_i shall be that of κ; the probability that in t series of v_1, v_2. v_t turns, each, the ϵ_i's shall never after v_j turns, all have the same sign as κ is

$$\prod_{j=1}^{j=t}\left(1 - \prod_{i=2}^{i=r} P_i^{(v_j)}\right),$$

and this quantity approaches zero as a limit as t increases indefinitely. But when all the ϵ's but ϵ_1 have the same sign as κ, the ruin of the one party or the other is even more certain than in the case of constant stakes.

The ruin of one party or the other being thus assured, we may easily calculate the probability favorable to each. Let P be the probability that the player shall ruin the banker, and $Q = 1 - P$ the probability that the banker shall ruin the player. As the game is fair, the expectation of each is zero, hence

$$Pn - Qm = 0, \qquad P = \frac{m}{m+n}, \qquad Q = \frac{n}{m+n}. \tag{2}$$

It should be noticed that these values are independent of the amounts staked.

We must now pass to the case of a game which is unfavorable to the player. Here

$$pb - qa < 0. \tag{3}$$

a and b shall be constant stakes, except in the extreme cases alluded to, where the one side or the other could not afford such a stake. Let us write the equation

$$f(x) \equiv px^{a+b} - x^a + q = 0,$$

$$f'(x) \equiv p(a+b)x^{a+b-1} - ax^{a-1}.$$

The roots of $f'(x)$ are

$$x = 0, \qquad x = \sqrt[b]{\frac{a}{p(a+b)}},$$

so that there is but one positive root. $f(x)$ has a root 1. Moreover

$$f'(1) = p(a+b) - a = pb - qa < 0,$$
$$f(\infty) = +\infty,$$

so that $f(x)$ has just one positive root, greater than unity. Call this α. Then

$$p\alpha^{a+b} - \alpha^a + q = 0, \qquad \alpha > 1. \tag{4}$$

We now make use of an ingenious device due to DeMoivre,* which consists in assuming that the player's fortune is composed of m counters whose values are α, α^2, $\cdots$ α^m while the banker has n counters α^{m+1}, α^{m+2}, $\cdots$ α^{m+n}. The player shall stake his a highest counters each time, and the banker his b lowest, so that when the lowest counter staked is α^x the player's expectation from the play is

$$p(\alpha^{x+a} + \alpha^{x+a+1} \cdots \alpha^{x+a+b-1}) - q(\alpha^x + \alpha^{x+1} \cdots + \alpha^{x+a-1})$$
$$= \frac{\alpha^x}{\alpha - 1}[p(\alpha^{a+b} - \alpha^a) - q(\alpha^a - 1)]$$
$$= \frac{\alpha^x}{\alpha - 1}[p\alpha^{a+b} - \alpha^a + q] = 0.$$

The game has thus been transformed into a fair one, the ruin of the one or the other is certain, and the probability favorable to each is equal to his fortune divided by the sum of the two fortunes. The player's chance is, thus,

$$\frac{\sum_{i=1}^{i=m} \alpha^i}{\sum_{i=1}^{i=m+n} \alpha^i} = \frac{\alpha^m - 1}{\alpha^{m+n} - 1}. \tag{5}$$

The banker's will be

$$\frac{\alpha^{m+n} - \alpha^m}{\alpha^{m+n} - 1}. \tag{6}$$

It seems to have escaped the notice of previous writers upon this subject that unless a and b are equal, and divide evenly into m and n, the values of α

* *The Doctrine of Chances*, London, 1756, p. 52.

or $m + n$ must be susceptible of modification in this formula, in order that it cover the case when one side or the other can not put up the requisite stake. This fact is theoretically important, but practically insignificant, as we have already pointed out. When the player's fortune has sunk to $\epsilon < a$, the chance favorable to him is certainly less than $\epsilon/(m + n)$, a trivial fraction, when n is large compared with a. On the other hand, when the banker's fortune is reduced to $\eta < b$, if the game be nearly fair, his chance is but little over $\eta/(m + n)$.

It is now time to take up the elusive case where the player may alter his stake at will. Assuming that a is the largest allowable stake, let us imagine that the player leads off with a stake a' less than a, but constantly plays a thereafter. Has he improved or injured his chance? The latter is, in the present case, made up of two parts, according as he wins or loses the first play, so that it has the value

$$p\,\frac{\alpha^{m+b'} - 1}{\alpha^{m+n} - 1} + q\,\frac{\alpha^{m-a'} - 1}{\alpha^{m+n} - 1}. \tag{7}$$

Subtracting this expression from (5) we get

$$\frac{\alpha^m - p\alpha^{m+b'} - q\alpha^{m-a'}}{\alpha^{m+n} - 1} = \frac{\alpha^m}{\alpha^{a'}(\alpha^{m+n} - 1)}\left(-p\alpha^{a'+b'} + \alpha^{a'} - q\right).$$

If we write $a' = a\rho$, $b' = \rho b$, $\rho < 1$ we see $\alpha' = \alpha^{\frac{1}{\rho}}$ is a root of the equation

$$px^{a'+b'} - x^{a'} + q = 0,$$

and

$$1 < \alpha < \alpha^{\frac{1}{\rho}},$$

$$p\alpha^{a'+b'} - \alpha^{a} + q < 0.$$

The difference between (7) and (5) is positive, or the probability favorable to the player has been lessened.

There is just one troublesome case where this reasoning is inapplicable, namely, where, after losing a' the player would have to reduce his stake, thus altering a in the second term of (7).

The player's stake, in case he lose the first play, shall now be

$$m - a' = \frac{a}{\sigma} \qquad \sigma > 1$$

We must consequently increase α to α^{σ}.

The player's chance thus becomes

$$p\,\frac{a^{m+b'}-1}{a^{m+n}-1}+q\,\frac{a^{\sigma(m-a')}-1}{a^{\sigma(m+n)}-1}\,. \tag{8}$$

It is necessary to compare the second term of (8) with that of (7). To do so let us note the alteration in the former as σ increases from 1. Expanding by Maclaurin's theorem we have

$$\frac{a^{\sigma(m-a')}-1}{a^{\sigma(m+n)}-1}=\frac{\Big[(m-a')\log a\Big]\sigma+\frac{1}{2\,!}\Big[(m-a')\log a\Big]^2\sigma^2+\cdots}{\Big[(m+n)\log a\Big]\sigma+\frac{1}{2\,!}\Big[(m+n)\log a\Big]^2\sigma^2\cdots}\,.$$

Let us write this

$$\frac{f(\sigma)}{\phi(\sigma)}=\frac{c\sigma+\frac{c^2}{2\,!}\sigma^2+\frac{c^3}{3\,!}\sigma^3+\cdots}{C\sigma+\frac{C^2}{2\,!}\sigma^2+\frac{C^3}{3\,!}\sigma^3+\cdots}, \qquad c<C$$

$$\frac{d}{d\sigma}\left[\frac{f(\sigma)}{\phi(\sigma)}\right]=\frac{\frac{cC}{2}\Big(c-C\Big)\sigma^2+\frac{cC}{3}\Big(c^2-C^2\Big)\sigma^3+\cdots}{\Big(C\sigma+\frac{C^2}{2\,!}\sigma^2+\cdots\Big)^2}\,.$$

This expression is negative for $\sigma>1$, hence (8) is less than (7), and, a fortiori less than (5).

We may pursue this line of reasoning further. As it was a mistake for the player to replace his first stake by another smaller stake, so it will be a further error for him to replace a in his second stake by a smaller value, and so on. We have already seen that any series of stakes whatsoever is sure to end in the ruin of the one party or the other, and we now see that no other series gives the player such a good chance as he gets by playing the maximum each time. We thus reach our fundamental theorem.

The player's best chance of winning a certain sum at a disadvantageous game, is to stake the sum which will bring him that return in one play, or, if that be not allowed, to make always the largest stake which the banker will accept.

The principal deduction from this is of a moral nature. The average gambler will say "The player who stakes his whole fortune on a single play is a fool, and the science of mathematics can not prove him to be otherwise." The reply is obvious: "The science of mathematics never attempts the impossible, it merely shows that other players are greater fools."

Let us make some simple deductions from our formula (5). The player's chance is always less than $m/(m+n)$. A better approximation, when m and n are both small is

$$\frac{m + \frac{1}{2} m(m-1)(a-1)}{(m+n) + \frac{1}{2}(m+n)(m+n-1)(a-1)}.$$

When m and n are equal the player's chance is

$$\frac{1}{a^m + 1}. \tag{9}$$

If the player should become frightened, and reduce his stake to $\frac{a}{\sigma}$, his chance would be reduced to

$$\frac{1}{a^{\sigma m} + 1}.$$

When a and b are equal

$$a = \frac{q}{p}. \tag{10}$$

When m is decidedly large, the probability that the banker will be ruined is close to

$$\frac{1}{a^n}. \tag{11}$$

3. The Game of Roulette. As an application of the foregoing principles, I am going to consider the game of roulette, as played at Monte Carlo. This game has two advantages, the honesty of the banker is above suspicion, and the player's disadvantage is less than in any other form of gambling with which I am familiar. The following description is taken from Sir Hiram Maxim.*

"The roulette consists of a large circular basin, about two feet in diameter, with the outer rim turned inward. The bottom of the basin, which forms the wheel,

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* *Monte Carlo, Facts and Fancies*, London, 1906, pp. 257 ff.

is of metal quite separate from the rim or sides, and is nicely balanced and mounted on a fine pivot so that when set in motion, it will spin for a considerable time. The outer edge of the wheel is accurately divided into thirty-seven sections of pockets, eighteen of which are painted red and eighteen black. One is called zero, and is neutral in color. The pockets are numbered from one to thirty-six." The wheel is set in motion, and a small ball started rolling around the edge in the opposite direction. The game consists in betting on the number or color of the division into which the ball will eventually roll. There are fourteen different methods of staking. The simplest are, red or black, even or odd, above or below eighteen. In each of these cases the player and banker stake equal sums. If a player stake upon a number, the banker puts up thirty-five times the amount. The upper limit for a simple stake, say red or black, is twelve hundred dollars; a player may not stake more than thirty-six dollars on a number. When the ball falls into zero, the player on a simple chance may either forfeit one half of his stake or leave the whole "in prison" till the next turn. If he be fortunate at this turn he saves his stake but gets nothing further, if he lose, the stake is gone forever. On the other hand all who have staked on particular numbers or combinations thereof lose their stakes at once on the arrival of zero.

What are the probabilities of success for a player staking on a simple chance as red or black? We may imagine that he plays twelve hundred dollars each time, and this sum we shall call one, so that m and n represent the player's and the banker's fortunes respectively, expressed as multiples of twelve hundred. In seventy-four turns the player may expect on an average, thirty-six reds, thirty-six blacks, one zero followed by a loss, and one zero followed by a gain, which leaves matters just as before and may be counted out. We have then

$$p = \frac{36}{73}, \quad q = \frac{37}{73},$$

$$a = b = 1, \quad \alpha = \frac{37}{36}.$$

The player's probability of winning is

$$\frac{\left(\frac{37}{36}\right)^m - 1}{\left(\frac{37}{36}\right)^{m+n} - 1}.$$

When m and n are equal this becomes

$$\frac{1}{\left(\frac{37}{36}\right)^m + 1}.$$

Let us next suppose that the player stakes upon a number. The amount which may be staked is, here, about one thirty-third of what it was before, so that the game is the same as if the stake remained constant, and the fortunes of both banker and player were increased thirty-three fold. The probability for the player will be the same as before if a be equal to the thirty-third root of $\left(\frac{37}{36}\right)$, and will be less if a be larger. Now the thirty-third root of $\frac{37}{36}$ is 1.00085.

To find a, we have $p = \frac{1}{37}$, $q = \frac{36}{37}$, $a = 1$, $b = 35$, $a = 1 + \epsilon$,

$$(1 + \epsilon)^{36} - 37(1 + \epsilon) + 36 = 0,$$

$$(1 + 36\,\epsilon + 630\,\epsilon^2) - 37 - 37\,\epsilon + 36 = 0,$$

$$1 + \epsilon = \frac{631}{630} = 1.0015.$$

This method is, therefore, somewhat less favorable to the player than the other.

A third simple game is played by staking on all the numbers but one. The maximum allowable here is thirty-five times thirty-six dollars, not far from twelve hundred, so that we may call the stake one as before. Each time the player wins at this game he sacrifices thirty-four thirty-fifths of his stake, so that b is one thirty-fifth. We shall further have

$$p = \frac{35}{37}, \quad q = \frac{2}{37},$$

$$qa - pb = \frac{1}{37},$$

whereas in the simple game

$$qa - pb = \frac{1}{73},$$

which shows that staking on thirty-five numbers would be an unwise method of play.

When n is large, the chance of ruin for the bank is

$$\frac{1}{a^n}.$$

We have no accurate data as to the amount of n, but apparently as the bank's earnings in a year are between five and six million dollars* it is not unreasonable to suppose that the sum of one million two hundred thousand dollars might be produced in a very short time. The chance of ruin on red or black is therefore not above

$$\left(\frac{36}{37}\right)^{1000.} = \frac{1}{792870000000}.$$

I conclude with two more quotations from Maxim's book. The first is from a letter from Baron Czyllak, quoted on p. 172.

"The calculation of probabilities by means of which inventors of systems seek to master roulette, and also the theory of roulette, have for their author the same savant that is to say, the celebrated Pascal." So does posterity treat the great theologian whose treatise on cycloids is called "Histoire de la roulette!"

The next quotation is from Maxim himself, on p. 190, and gives the best explanation which I have seen for the fact the people continue to gamble:

"Je me rends parfaitement compte du désagreable effet que produit sur la majorité de l'humanité, tout ce qui se rapporte, même au plus faible dégré, à des calculs ou raisonnements mathématiques."

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* *Maxim*, loc. cit. p. 9.